Optimal Weighted Data Gathering in Multi-Hop Heterogeneous Sensor Networks

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***Abstract*—We consider the problem of gathering data from an energy-limited wireless sensor network consisting of a set of sensor nodes and a sink node. Based on the heterogeneous nature of sensor networks, where data generated at different sensor nodes are of different fidelity and importance, we formulate the data gathering problem as maximizing the weighted sum of data generated at all sensors and turn it to a linear programming problem subject to the limited-energy constraints. Furthermore, we show that the optimal multi-hop routing for the weighted data gathering problem is *character-based routing*, where only "bad" sensors relay data for "good" sensors. An example application and simulation results are given to justify the proposed concept and algorithm.**

I. Introduction

Wireless sensor networks (WSN), consisting of a large number of geographically distributed sensor nodes, have many current and future envisioned applications, such as environment monitoring, battlefield surveillance, health care, and home automation [1]. Though each sensor is characterized by low power constraint and limited computation and communication capabilities, potentially powerful networks can be constructed to accomplish various high-level tasks via sensor cooperation [2], such as distributed estimation, distributed detection, and target localization and tracking.

A common goal in most WSN applications is to continuously monitor the sensor field or objects, and transmit the sensed data to the sink node (base station) for further processing, such as inferencing the environmental parameters and events of interest or reconstructing the underlying physical phenomenon. To maximize the network usage, it is desired to extract as much data as possible from the network before it depletes. Thus, in *data gathering problem*, the objective is to maximize the amount of data gathered from all sensors to the sink node using direct or multi-hop radio transmission, under the limited energy constraints of all sensors.

Several papers have addressed the data gathering problem in energy limited wireless sensor networks [3]–[5]. The work of [3] presents an optimal framework for the maximal data extraction problem to incorporate different constraints, such as flow rate, transmission power and fairness. In [4], the authors propose an approximate algorithm to route the information for maximum data extraction problem. In [5], an efficient and implementable algorithm for maximizing data extraction problem, which focuses on the transient behavior of protocols in particular whether they exhibit fast convergence. However, most of the works formulate the data gathering problem in terms of maximizing the total quantity of the data gathered, no matter how heterogeneous the network may be, where the data from different sensor may have very different fidelity and importance.

To explicitly take the heterogeneity nature of sensor networks into account, we consider the *weighted data gathering* problem, which treats the data from different sensors unequally. Using multi-hop routing to transmit data from all sensors to the sink node, which is essential to reduce the transmission energy cost and thus increase the gathered data volume, the problem can be formulated as a linear programming (LP) problem of network flow. Furthermore, the optimal multi-hop routing is shown to be *character-based routing*, where only "bad" sensors relay data for "good" sensors[1], which is different from the traditional distance-based routing, where sensor nodes closer to the sink node relay data for sensor nodes farther away from the sink node.

The rest of the paper is organized as follows. Section II introduces the weighted data gathering problem in wireless sensor networks using parameter estimation as an example, and formulates it as a linear program-

[1]Here, "good" sensors denote those sensors whose data are more accurate and more important than others, thus bigger weight factors are assigned for them, vice versa for "bad" sensors.

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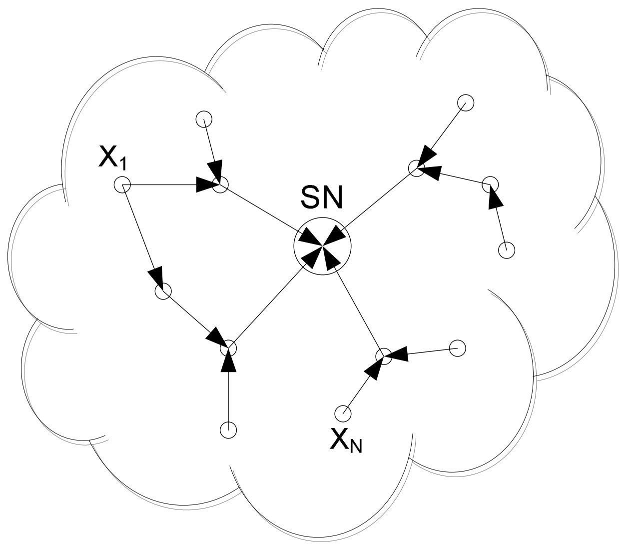

Fig. 1. An example of a wireless sensor network with N distributed sensor nodes. Each sensor makes observations on the sensor field and sends its sensed data to the sink node (SN) for further processing. In directed solid lines, a chosen multi-hop routing path is shown, where the data from a sensor can be relayed by multiple sensors, meanwhile a sensor can relay data for multiple sensors.

ming (LP) problem. Section III introduces the notion of character-based routing and shows that the optimal multi-hop routing for the given weighted data gathering problem is character-based routing. In Section IV, we show some simulation results. Finally, conclusions are given in Section V.

II. Weighted Data Gathering

We consider a dense sensor network including N distributed sensor nodes and a sink node (denoted as node $N+1$), as shown in Fig. 1. In a heterogeneous sensor networks, different sensors have different characteristic, i.e., different fidelity and different importance, thus it is necessary to treat the data generated at different sensors unequally. In weighted data gathering (WDG) problem, the objective is to maximize the weighted sum of the data gathered from all sensors to the sink node, i.e.,

$$\text{maximize} \quad \sum_{k=1}^{N} w_k S_k, \tag{1}$$

where, S_k denotes the amount of data in bits generated by sensor k $(k = 1, \cdots, N)$, and w_k denotes the weight for the data of sensor k.

A. An Example of WDG - Parameter Estimation

To justify the notion of weighted data gathering, we use the parameter estimation problem in wireless sensor networks as an example. In the estimation of an unknown parameter with a sensor network consisting of N sensor nodes and a sink node, each sensor observes the parameter θ and transmits its observations to the sink node, which makes the final estimation based on the received messages.

First, the observations of all sensors are corrupted by additive noise and described by

$$x_k = \theta + n_k, \quad k = 1, \cdots, N. \tag{2}$$

The observation noises of all sensors n_k $(k = 1, \cdots, N)$ are assumed to be zero mean, spatially uncorrelated with variance σ_k^2, while the noise at each sensor is assumed to be temporally i.i.d distributed, otherwise unknown.

Second, assume there are K observations $(x_1, \cdots, x_K)$ available at the sink node, then the sink node can make an estimation using best linear unbiased estimator (BLUE) [6] as follows

$$\bar{\theta} = \left(\sum_{k=1}^{K} \frac{1}{\sigma_k^2} \right)^{-1} \sum_{k=1}^{K} \frac{x_k}{\sigma_k^2}, \tag{3}$$

and the estimation MSE of the BLUE estimator is

$$E(\bar{\theta} - \theta)^2 = \left(\sum_{k=1}^{K} \frac{1}{\sigma_k^2} \right)^{-1}. \tag{4}$$

From application aspect, the network is considered functional if it can produce an estimation $\bar{\theta}$ satisfying a distortion requirement D_r, i.e., $E(\bar{\theta} - \theta)^2 \leq D_r$, otherwise it is nonfunctional. Then we define the network lifetime L as the estimation task cycles accomplished before the network becomes nonfunctional due to sensor depletion.

At different estimation cycles, the parameter θ is assumed to be unrelated, and the estimation at each cycle is performed independently using only the observations made by all sensors in the given estimation cycle. To satisfy the given estimation distortion requirement D_r, at each estimation cycle, a subset of sensors are required to observe the parameter θ and transmit their quantized measurements to the fusion center to make the final estimation. Assume sensor k $(k = 1, \cdots, N)$ can make a total of M_k measurements before it depletes, then as shown in Appendix A, the network lifetime L for estimation is bounded as

$$L \leq D_r \left(\sum_{k=1}^{N} \frac{M_k}{\sigma_k^2} \right). \tag{5}$$

Therefore, for parameter estimation application, maximizing the network lifetime for a given estimation distortion requirement D_r is equivalent to a weighted data gathering problem stated as

$$\text{maximize} \quad \sum_{k=1}^{N} \frac{M_k}{\sigma_k^2}, \tag{6}$$

where, $w_k = 1/\sigma_k^2$ is the weight for sensor k, that is to say, the weight w_k for sensor k is inversely proportional to its observation noise variance. This is a meaningful weighting rule since different observation noise variances mean different levels of accuracy, more specifically, smaller observation noise variance means higher level of accuracy.

Though we use parameter estimation as an example of weighted data gathering problem, it is worth pointing out that the weighted data gathering problem in Eq. (1) is a generic framework to address the heterogeneity of sensor networks by intelligently designing the weighting rule for different applications.

B. Linear Programming (LP) Formulation of WDG

In energy-limited wireless sensor networks, multi-hop transmission is essential to save transmission energy and thus increase the gathered data volume. In multi-hop wireless sensor networks, each sensor not only transmits the data generated by itself, but also relays the data for other sensors. Since the total amount of data each sensor can transmit and relay is limited by the energy supply of the sensor node, the amount of data generated at each sensor and the multi-hop routing path from each sensor to the sink node need to be optimized together to maximize the weighted data gathering in Eq. (1).

Assume sensor nodes can adjust their transmission power to control the transmission range. The energy consumed by sensor i to reliably transmit b-bit information to sensor j is

$$e(b) = c \cdot b \cdot d_{i,j}^{\alpha}, \tag{7}$$

where c is a system constant, α is the path loss exponent, and $d_{i,j}$ is the distance between sensor i and sensor j.

Model the wireless sensor network as a directed graph $G(V, E)$, where V is the set consisting of all the N sensor nodes and sink node (node $N+1$), i.e., $V = [1, N+1]$, E is the set of directed links in the network. An edge $(i, j) \in E$ iff $d_{i,j} \le R$, where R is the maximum transmission range. The link cost, denoted as $C_{i,j}$, to transmit a unit bit information from node i to node j depends on the distance $d_{i,j}$ between them based on the energy model in Eq. (7) as follows,

$$C_{i,j} = \begin{cases} c d_{i,j}^{\alpha}, & \text{if } d_{i,j} \le R \\ +\infty, & \text{otherwise} \end{cases} \tag{8}$$

Subject to the limited energy supply P_k of each sensor k $(k = 1, \cdots, N)$, the weighted data gathering problem can be formulated as a linear programming (LP) problem as follows:

$$\text{maximize} \quad \sum_{k=1}^{N} w_k S_k, \tag{9}$$

subject to

$$\sum_{i=1, i\neq k}^{N} f_{i,k} + S_k = \sum_{j=1, j\neq k}^{N+1} f_{k,j}, \quad \forall k \in [1, N] \tag{10}$$

$$\sum_{j=1, j\neq k}^{N+1} f_{k,j} C_{k,j} \leqslant P_k, \quad \forall k \in [1, N] \tag{11}$$

where

$$\begin{array}{ll} S_k \ge 0, & \forall k \in [1, N] \\ f_{i,j} \ge 0, & \forall i \in [1, N], j \in [1, N+1] \end{array} \tag{12}$$

w_k is the weight of sensor node k and is given, P_k is the total energy supply and is also given. S_k denotes the total amount of data generated at sensor node k, and $f_{i,j}$ denotes the amount of data transmitted from sensor node i to sensor node j (there is no outcoming flow from the sink node). Both S_k and $f_{i,j}$ are variables to be optimized to maximize the gathered data volume. Eq. (10) and Eq. (11) represent two constraints of the optimization problem:

1) *flow conservation*: the amount of data transmitted by a sensor node is equal to the sum of the amount of data received by the sensor node and the amount of data generated by the sensor node itself.
2) *energy constraint*: the amount of data transmitted by a sensor node is limited by the energy supply of the sensor node.

Furthermore, it is easy to extend the linear programming model above to a more general case, where there exists a third type of sensor nodes, which only can act as relay without observation capabilities, simply by assigning a weight factor 0 for those relay nodes.

III. Character-based Routing

As shown in Section II-B, the optimal weighted data gathering problem is formulated as a linear programming problem, thus it can be easily solved by any LP solver, such as [7], which is used in our simulations. It is interesting to note that, in the optimal multi-hop routing structure for this problem, a sensor node only relays data generated by sensor nodes with higher importance, i.e., bigger weight, as shown in Theorem 1. That is to say, the optimal routing is based on the character (fidelity and importance) of the sensor nodes, thus it is called *character-based routing*. Character-based routing is a new notion for routing and it is different from the traditional distance-based routing, such as shortest path tree, where a sensor node closer to the sink node relays information for sensor nodes farther away from the sink node.

Theorem 1 *The optimal routing structure for the weighted data gathering problem shown in Eq. (9, 10, 11) is character-based routing, where a sensor node only relays data generated by sensor nodes with higher importance, i.e., bigger weight. More specifically, in the optimal flow and routing solution, let η be a sub flow with data volume S, generated at sensor node i_0 and relayed by sensor nodes $i_1, \cdots, i_T$ sequentially to the sink node, i.e.,*

$$S_{i_0}^{\eta} = f_{i_0,i_1}^{\eta} = f_{i_1,i_2}^{\eta} = \cdots = f_{i_{T-1},i_T}^{\eta} = f_{i_T,N+1}^{\eta} = S, \tag{13}$$

then,

$$w_{i_0} \geq w_{i_t}, \quad \forall t \in [1, T]. \tag{14}$$

Proof: We prove it by contradiction. Assume a sensor i_m ($m \in [1, T]$) on the routing path of the sub flow η has a bigger weight factor than the source node i_0, i.e.,

$$w_{i_m} > w_{i_0}, \tag{15}$$

then, remove the sub flow η and add a new sub flow ξ with the same data volume, which is generated at sensor node i_m and transmitted to the fusion center through sensor nodes $i_{m+1}, \cdots, i_T$ sequentially, i.e.,

$$\begin{array}{l} S_{i_0}^{\eta} = f_{i_0,i_1}^{\eta} = f_{i_1,i_2}^{\eta} = \cdots = f_{i_{T-1},i_T}^{\eta} = f_{i_T,N+1}^{\eta} = 0, \\ S_{i_m}^{\xi} = f_{i_m,i_{m+1}}^{\xi} = \cdots = f_{i_{T-1},i_T}^{\xi} = f_{i_T,N+1}^{\xi} = S. \end{array} \tag{16}$$

First, it is easy to show that both the flow conservation and energy constraints as shown in Eq. (10, 11) are satisfied by removing the sub flow η and adding the new sub flow ξ, that is to say, the new network flow is feasible.

Next, assume the total data volume generated at each sensor k is S_k and denote ϕ_0 and ϕ_1 as the objective function before or after removing the sub flow ξ and adding the new sub flow τ, respectively, i.e.,

$$\begin{aligned} \phi_0 &= \sum_{k=1}^{N} w_k S_k, \\ \phi_1 &= \sum_{k=1, k \neq i_0, i_m}^{N} w_k S_k + w_{i_0}(S_{i_0} - S) + w_{i_m}(S_{i_m} + S), \end{aligned} \tag{17}$$

then,

$$\phi_1 - \phi_0 = (w_{i_m} - w_{i_0})S > 0. \tag{18}$$

It means that the objective function in Eq. (9) is increased by removing the sub flow η and adding the new sub flow ξ, which contradicts with the optimality of the original flow and routing solution. So the assumption made in Eq. (15) does not hold, therefore, the Theorem is proved. ■

As a special case of weighted data gathering in heterogeneous wireless sensor networks, in the following Proposition 1, we consider the equally weighted data gathering in a homogeneous wireless sensor network, where all sensors have the same characteristics, i.e., same fidelity and importance.

Proposition 1 *In homogeneous wireless sensor networks, the weighted data gathering problem shown in Eq. (9, 10, 11) retrogresses to an equally weighted data gathering problem, i.e., $w_k = w$ ($k = 1, \cdots, N$), and the corresponding optimal routing solution is single-hop routing, i.e., all sensors transmit their data to the sink node directly.*

Proof: We prove it by contradiction. In the optimal flow and routing solution for the weighted data gathering problem in homogeneous wireless sensor networks, assume there is a multi-hop sub flow η with data volume S, generated at sensor i_0 and transmitted to the fusion center through sensors $i_1, \cdots, i_T$ sequentially, i.e.,

$$S_{i_0}^{\eta} = f_{i_0,i_1}^{\eta} = f_{i_1,i_2}^{\eta} = \cdots = f_{i_{T-1},i_T}^{\eta} = f_{i_T,N+1}^{\eta} = S \tag{19}$$

then, remove this multi-hop sub flow η and add a serial of single-hop sub flow $\xi_0, \cdots, \xi_T$ as follows:

$$\begin{array}{l} S_{i_t}^{\xi_t} = f_{i_t,N+1}^{\xi_t} = \frac{C_{i_t,i_{t+1}}}{C_{i_t,N+1}} \cdot S, \quad \forall t \in [0, T-1] \\ S_{i_T}^{\xi_T} = f_{i_T,N+1}^{\xi_T} = S. \end{array} \tag{20}$$

Similar with the proof for Theorem 1, it is easy to show that both the flow conservation and energy constraints as shown in Eq. (10, 11) are satisfied by removing the sub flow η and adding the new sub flows $\xi_0, \cdots, \xi_T$.

Next, assume the total data volume generated at each sensor k is S_k and denote ϕ_0 and ϕ_1 as the objective function before or after removing the sub flow η and adding the new sub flows $\xi_0, \cdots, \xi_T$, i.e.,

$$\begin{aligned} \phi_0 &= w \sum_{k=1}^{N} S_k, \\ \phi_1 &= w \left(\sum_{k=1, k \neq i_0, i_T}^{N} S_k + (S_{i_0} - S) + (S_{i_T} + S) \right) \\ &\quad + w \sum_{t=0}^{T-1} \frac{C_{i_t,i_{t+1}}}{C_{i_t,N+1}} \cdot S, \end{aligned} \tag{21}$$

then,

$$\phi_1 - \phi_0 = w \sum_{t=0}^{T-1} \frac{C_{i_t, i_{t+1}}}{C_{i_t, N+1}} \cdot S \geq 0, \tag{22}$$

where, the equality holds only when the fusion center is not in the transmission range of all sensors $i_0, \cdots, i_{T-1}$, i.e., $C_{i_t,N+1} = \infty$ for all $t \in [0, T-1]$, otherwise, $\phi_1 - \phi_0 > 0$. It means that for homogeneous networks with unlimited transmission range for each sensor, single-hop routing can gather greater amount of data than multi-hop routing, while for homogeneous networks with limited transmission range for each sensor, single-hop routing can gather no less amount of data than multi-hop routing. ■

IV. Simulation Results

In this section, we simulate a heterogeneous sensor network, and the parameter estimation problem introduced in Section II-A is used as an example of the proposed weighted data gathering problem.

Assume there are N sensors, and the observation noises of each sensor is assumed to be

$$\sigma_k^2 = \beta + \gamma z_k, \quad k = 1, \cdots, N, \tag{23}$$

where β models the network-wide noise variance threshold, γ controls the underlying variation from sensor to sensor, and $z_k \sim \chi_1^2$ is a Chi-Square distributed random variable with one degree of freedom. It is noted that the network is homogeneous for the special case of $\gamma = 0$. In the experiments, we assume $\beta = 0.01$ and $\gamma = 0.00, 0.05, 0.10, 0.15,$ or 0.20. Assume all sensors are independently and uniformly distributed in a rectangular region of $[-5, 5, -5, 5]$, and the sink node is located at the central point of the region, i.e., $(0, 0)$. And the initial energy supply is assumed to be the same for all sensors.

For a given network setting, the optimal data volume generated at each sensor and the multi-hop routing solutions are determined to maximize the weighted data gathering. To demonstrate the efficiency of the optimal multi-hop routing, we compare it with a heuristic method - single-hop routing, which is the optimal solution without considering the heterogeneity of the networks as shown in Proposition 1.

Fig. 2 shows the ratio of the amount of data gathered by the proposed algorithm to the heuristic method under different total number of sensors and different sensor noise variation parameters γ, where all the simulation results are obtained by repeating the experiments for 2000 times and averaging the individual results. It is worth noting that, for the estimation problem introduced in Section II-A, the volume of data gathered is equivalent to the sensor network lifetime according to Eq. (5).

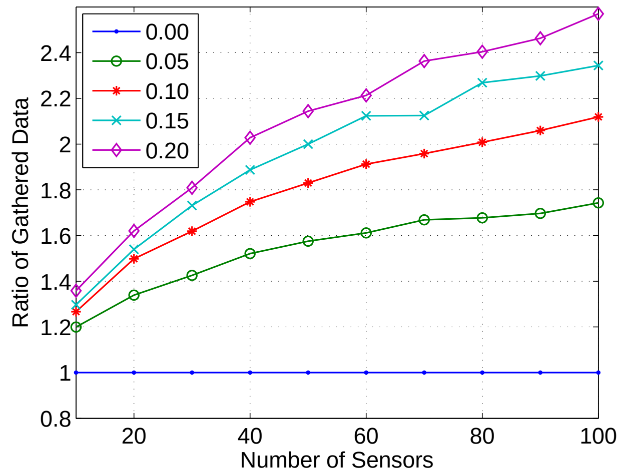

Fig. 2. Ratio of data volume gathered by the proposed algorithm to the heuristic method under different total number of sensors and different sensor noise variation parameters ($\gamma = 0.00, 0.05, 0.10, 0.15, 0.20$).

From Fig. 2, we see that the heuristic method is also optimal when $\gamma = 0.00$, which confirms our conclusion in Proposition 1. From Fig. 2, we also can see that optimal multi-hop routing increases the gathered data volume (thus the network lifetime) significantly compared with single-hop routing for heterogeneous networks. Furthermore, the gain is more significant when the network is denser since there are more opportunities for multi-hop routing, also the gain is more significant when the observation noise variances are more diverse, i.e., γ becomes bigger, since the optimal multi-hop routing is character-based as shown in Section III.

To further demonstrate the character-based routing, Fig. 3(a) and Fig. 3(b) show two example heterogeneous sensor networks with $N = 10$ sensor nodes, where each circle denotes a sensor node. There are two numbers in the brackets around each sensor node, where the first one denotes its index and the second one denotes its observation noise variance. In these two networks, the sensor locations are the same, while the observation noise variances are different. From Fig. 3(a) and Fig. 3(b), we can see that the optimal routing completely changed due to the different observation noise variances, and the sensors only relay information generated at other sensors with smaller observation noise variances, such as in Fig. 3(a), sensor 8 relays information from sensor 3, while in Fig. 3(b), sensor 3 relays information from sensor 8 even though sensor 3 is farther away from the fusion center than sensor 8. The intuitive explanation is that sensor 8 has a very small observation noise variance,

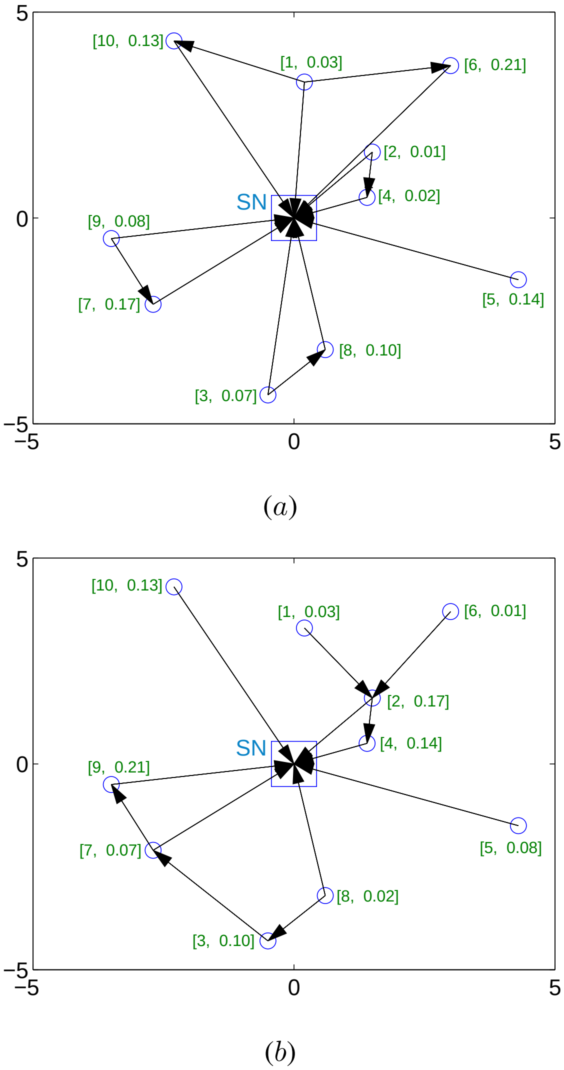

Fig. 3. The optimal multi-hop routing path for two example heterogeneous sensor networks with same sensor locations but different observation noise variances.

then it is desirable to gather as much data as possible from sensor 8, thus sensor 8 should transmit its data to its nearest neighbor (sensor 3) if possible to save transmission energy and improve source throughput. It is also noted that in Fig. 3(b), sensor 7 relays some information from sensor 3 even though sensor 7 has smaller observation noises than sensor 3 because the relayed information is originally generated at sensor 8 other than sensor 3.

V. Conclusions

In this paper, we consider the weighted data gathering problem in energy-limited wireless sensor networks to maximize the weighted sum of the amount of data generated at all sensors, where the weight accounts for the heterogeneity of the networks. We formulate the weighted data gathering problem as a linear programming (LP) problem, and find out that the optimal multi-hop routing solution is *character-based routing*, where a sensor node only relays data generated at sensor nodes with higher importance, i.e., bigger weight. Different from the traditional distance-based routing, where the routing path is selected based on the distance to the destination, *character-based routing* explicitly takes the heterogeneity nature of the information in networks into account in the routing selection.

Further generalization of the new notion of character-based routing is an interesting direction for the future work. A distributed implementation of the character-based routing to achieve the optimal weighted data gathering is another desirable future work.

Appendix A
Proof of Equation (5)

Assume a sensor network with N sensors, each with observation noise variance σ_k^2. Assume sensor k ($k = 1, \cdots, N$) can make a total of M_k measurements before it depletes. To satisfy the given estimation distortion requirement D_r, at each estimation cycle, a subset of sensors are required to observe the parameter θ and transmit their measurements to the fusion center to make the final estimation.

Assume the network lifetime for this network is L. At each estimation cycle $l \in [1, L]$, denote the subset of observations each sensor k makes and sends to the fusion center is $O_{k,l}$. Then for any sensor $k \in [1, N]$, we have

$$\begin{aligned} & O_{k,i} \cap O_{k,j} = \emptyset, \quad \forall i, j \in [1, L], \text{ and } i \neq j, \\ & \bigcup_{l=1}^{L} O_{k,l} \subseteq \{1, \cdots, M_k\}, \end{aligned} \tag{24}$$

and for any estimation cycle $l \in [1, L]$, we have

$$\left(\sum_{k=1}^{N} \sum_{i \in O_{k,l}} \frac{1}{\sigma_k^2} \right)^{-1} \leq D_r. \tag{25}$$

So,

$$\sum_{l=1}^{L} \sum_{k=1}^{N} \sum_{i \in O_{k,l}} \frac{1}{\sigma_k^2} \geq \frac{L}{D_r}, \tag{26}$$

i.e.,

$$\sum_{k=1}^{N} \sum_{i=1}^{M_k} \frac{1}{\sigma_k^2} \geq \frac{L}{D_r}, \tag{27}$$

therefore,

$$L \leq D_r \left(\sum_{k=1}^{N} \sum_{i=1}^{M_k} \frac{1}{\sigma_k^2} \right). \tag{28}$$

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